PISTON MOTION UNDER THE EFFECT OF GAS PRESSURE

(DVIZHENIE PORSHNIA POD DEISTVIEM DAVLENIIA GAZA)

PMM Vol.27, No.3, 1963, pp. 535-540

Iu.A. SOZONENKO
(Moscow)

(Received February 8, 1963)

The problem of the motion of a piston struck by an expanding gas with adiabatic index 11/9 was solved by series by Love [1].

For any adiabatic index γ this problem is reduced herein to the solution of an ordinary nonlinear differential equation.

An exact Bessel function solution of that differential equation is obtained for the case $\gamma = 1.67$ (monatomic gas).

1. A gas at rest, whose pressure is p_0, density ρ_0 and speed of sound a_0, is between two clamped pistons in an infinite tube of constant cross-section. The pressure is zero outside the space between the pistons. The mass of each piston is m. The pistons are released at time $t = 0$. The subsequent motion of the pistons under the effect of the gas pressure is studied. The gas will be assumed perfect, inviscid and non-conducting. The friction of the piston on the tube walls is neglected.

Let h be a Lagrange coordinate; $u(h, t)$ an Euler coordinate; γ the adiabatic index. The pistons have the coordinates $h = 0$ and $h = 2l$; the tube cross-sectional area is F. The equation of the gas flow between the pistons is

$$\frac{\partial^2 u}{\partial t^2} = a_0^2 \left(\frac{\partial u}{\partial h}\right)^{-\gamma-1} \frac{\partial^2 u}{\partial h^2} \qquad (-2l \leqslant h \leqslant 0) \tag{1.1}$$

For $t = 0$ we have the initial conditions

$$u(h, 0) = h, \qquad \frac{\partial u}{\partial t} = 0$$

As boundary conditions we have

$$m\frac{\partial^2 u}{\partial t^2} = Fp_0\left(\frac{\partial u}{\partial h}\right)^{-\gamma} \quad \text{where } h = 0$$
$$m\frac{\partial^2 u}{\partial t^2} = -Fp_0\left(\frac{\partial u}{\partial h}\right)^{-\gamma} \quad \text{where } h = -2l \qquad (1.2)$$

Let us introduce nondimensional quantities by means of the formulas:

$$x = \frac{h}{l}, \qquad \tau = \frac{(\gamma - 1)\,F\rho_0 a_0}{2\gamma m}\,t, \qquad \sigma = \frac{a}{a_0}$$
$$v = \frac{\gamma - 1}{2a_0}\frac{\partial u}{\partial t}, \qquad \theta = \frac{u}{l} \qquad (1.3)$$
$$\mu = \frac{\gamma - 1}{\gamma}\frac{\rho_0 F l}{m} = \frac{\gamma - 1}{\gamma}\frac{m_0}{m}$$

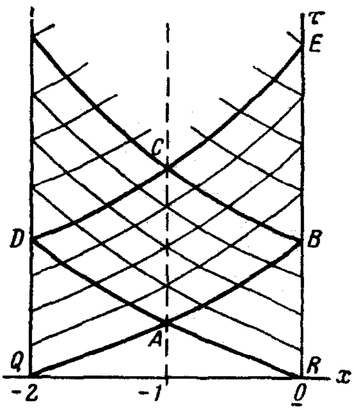

Fig. 1.

Let us note that only two nondimensional constants μ and γ enter into the initial data, boundary conditions and equation of motion. Here μ is defined by the adiabatic index and the ratio of the mass of the gas to the mass of the piston; m_0 is the mass of gas in the volume $l \leqslant h \leqslant 0$. In the new variables the pistons have the coordinates $x = 0$ and $x = -2$. The diagram of the flow in the x, τ plane is given in Fig. 1. As is known, at the initial instant, simple waves proceed from the pistons toward each other (the zones *ARB* and *AQD* in Fig. 1). After meeting, the simple waves start to interact (zone *ABCD* of Fig. 1). After interaction, the transmitted wave is reflected from the piston (zone *BCE* of Fig. 1). The wave interaction and reflection from the piston are duplicated in the subsequent motion. In the simple wave zone the velocity and coordinate of the piston $x = 0$ are determined from the formulas [1, 2]:

$$v = 1 - (1 + 2n\tau)^{-1/n}, \qquad \theta = \frac{(1 - v)^{2n} - 1 + 2nv}{n\mu\,(1 - v)^{2n}}, \qquad n = \frac{\gamma + 1}{2\,(\gamma - 1)} \qquad (1.4)$$

which are valid up to the instant when wave reflection from the piston starts (point *B* in Fig. 1).

The velocity, time and coordinate of the piston corresponding to the point *B* are

$$v_* = 1 - (1 + n\mu)^{-1/n}, \qquad \tau_* = {}^1/_2\ \mu\,(2 + n\mu) \qquad (1.5)$$
$$\theta_* = \frac{1 + (2n - 1)\,(1 + n\mu)^2 - 2n\,(1 + n\mu)^{2-1/n}}{n\mu}$$

The problem of simple wave interaction (zone *ABCD* in Fig. 1) is considered [1-5]. A solution for the zone of reflection from the piston (BCE in Fig. 1) is given below. Let us introduce nondimensional Riemann variables by means of the formulas:

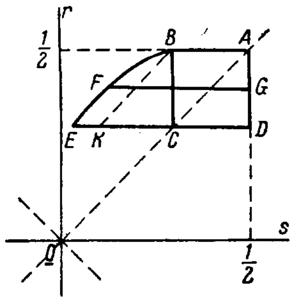

Fig. 2.

$$v+\left(\frac{\partial\theta}{\partial x}\right)^{-(\gamma-1)/2}=2r,\quad v-\left(\frac{\partial\theta}{\partial x}\right)^{-(\gamma-1)/2}=-2s \qquad (1.6)$$

From (1.6)

$$v=r-s,\qquad \frac{\partial\theta}{\partial x}=(r+s)^{-2/(\gamma-1)},\qquad \sigma=r+s$$

The diagram of the flow in the r, s plane is depicted in Fig. 2. The rest zone ARQ is mapped in the r, s plane by the point $A(1/2,\ 1/2)$. Simple waves are mapped in the r, s plane by the lines AB and AD (Fig. 2). The zone of simple-wave interaction is mapped in Fig. 2 by the square $ABCD$.

The point $B(1/2,\ 1/2-v_*)$ in Fig. 2 corresponds to the beginning of the first reflection from the piston $x = 0$; the point $D(1/2-v_*,\ 1/2)$ is the beginning of the reflection from the piston $x = -2$. The zone of reflection of the transmitted wave from the piston $x = 0$ is mapped in the r, s plane by the curvilinear triangle BEC. The curve BE will be the image of the piston $x = 0$ in the r, s plane.

2. Let us show that the number of wave reflections from the piston is finite and let us find the upper bound of this number. Let us draw the line $r - s = v_*$ (the segment KB) in Fig. 2. It is seen from the sketch that the point K falls on the line $r + s = 0$ at $v_* = 1/3$. Actually, in this case the point B has the coordinates $(1/2,\ 1/6)$ and we have the coordinates of the point $K(1/6,\ -1/6)$ from the equality $BC = KC$. The curve BE which always lies above the curve KB certainly intersects the line $r + s = 0$ in the zone of the first reflection from the piston in this case. But the intersection of the curve mapping the piston with the line $r + s = 0$ denotes the achievement of zero speed of sound and zero pressure at the piston. This is only possible in the limit as $\tau \to \infty$. Therefore, only one reflection from the piston occurs for $v_* \geqslant 1/3$. The characteristic CE in Fig. 1 does not intersect the line $x = 0$ for $v_* \geqslant 1/3$ (the characteristic does not reach the piston). The possibility of such a phenomenon was first noted by Staniukovich [2]. Expressing v_* from (1.5), we obtain the values of μ for which only one reflection from the piston occurs:

$$\mu_1 \geqslant \frac{1}{n}\left[\left(\frac{3}{2}\right)^n - 1\right]$$

In an analogous manner it is easy to obtain the general formula for the values of μ for which not more than k reflections from the piston occur:

$$\mu_k \geqslant \frac{1}{n}\left[\left(\frac{2k+1}{2k}\right)^n - 1\right]$$

3. It was shown [6] that with the exception of the simple-wave zone and the zone of constant solutions, the Riemann equation

$$L[x] = \frac{\partial^2 x}{\partial r\, \partial s} - \frac{n}{r+s}\left(\frac{\partial x}{\partial r} + \frac{\partial x}{\partial s}\right) = 0 \tag{3.1}$$

is equivalent to equation (1.1).

The function $\tau(r, s)$ satisfies an analogous equation and can also be determined from the relations

$$\frac{\partial x}{\partial r} = -\frac{2}{\mu}(r+s)^{2n}\frac{\partial \tau}{\partial r}, \qquad \frac{\partial x}{\partial s} = \frac{2}{\mu}(r+s)^{2n}\frac{\partial \tau}{\partial s} \tag{3.2}$$

Let us formulate the boundary conditions for equation (3.1) in the square $ABCD$ and the curvilinear triangle BEC (Fig. 2).

The conditions on the line AB can be obtained from the solution for the simple wave (Fig. 1) in the zone ARB.

From (3.2) we have correspondingly on the characteristic AB

$$\frac{\partial x}{\partial s} = \frac{2}{\mu}\left(\frac{1}{2}+s\right)^{2n}\frac{\partial \tau}{\partial s}, \qquad \frac{x}{\tau-\tau_1} = -\frac{2}{\mu}\left(\frac{1}{2}+s\right)^{2n} \tag{3.3}$$

Here τ_1 is the value of τ at the intersection of the characteristic with $x = 0$. Eliminating τ and using (1.3), we have on AB and AD, respectively, (on AD by analogy):

$$\frac{\partial x}{\partial s} - \frac{nx}{1/2+s} = -\frac{1}{\mu}\frac{1}{1/2+s}, \qquad \frac{\partial x}{\partial r} - \frac{n(x+2)}{r+1/2} = \frac{1}{\mu}\frac{1}{r+1/2} \tag{3.4}$$

Solving (3.4), we obtain

$$x(s) = -\left(\frac{1}{2}+s\right)^n + \frac{1}{n\mu}\left[1-\left(\frac{1}{2}+s\right)^n\right] \quad \text{on } AB \tag{3.5}$$

$$x(r) = -2 + \left(r+\frac{1}{2}\right)^n - \frac{1}{n\mu}\left[1-\left(r+\frac{1}{2}\right)^n\right] \quad \text{on } AD \tag{3.6}$$

The condition on the curve BE is:

$$x(r, s) = 0 \tag{3.7}$$

In the r, s variables the piston-motion equation (1.2) is

$$\frac{dv}{d\tau} = (r+s)^{2n+1} \tag{3.8}$$

The relation (3.7) is the boundary condition for (3.1) on the unknown curve *BE*. Equation (3.8) is used to find the curve *BE*. Conditions (3.7) and (3.8) can be represented in more symmetric form. We have on *BE* from (3.2), (3.7), (3.8):

$$\frac{\partial x}{\partial r} = -\frac{1}{\mu}(r+s)^{-1}\frac{dv}{dr}, \qquad \frac{\partial x}{\partial s} = \frac{1}{\mu}(r+s)^{-1}\frac{dv}{ds} \tag{3.9}$$

Hence, the boundary conditions for $x(r, s)$ on the two intersecting characteristics *AB* and *AD* (Goursat problem) are given for (3.1) in the square *ABCD* by (3.5), (3.6). The solution of this problem for (3.1) with the boundary conditions (3.5), (3.6) is given in [1] by the Riemann method [6]. The values of $x(r, s)$ on the characteristic *BC* can be found from this solution. Then the values of $x(r, s)$ on the time-similar curve *BE* and the characteristic *BC* (mixed problem) will be given in the curvilinear triangle *BEC*. The existence and uniqueness of the solution of the mixed problem for linear equations was proved by Goursat [7] by successive approximations. Hadamard showed that the Riemann method can be extended to the mixed problem (see [8] for example). The solution of equation (3.1) under conditions (3.7), (3.8) is given in series by Love [1]. But the numerical computations in that paper are carried out only for $n = 5$, $\mu = 0.04364$. Because of the awkwardness of numerical computations for other values of μ and n no others were carried out. A method of solving (3.1) under conditions (3.7), (3.8) based on the use of the Green's formula and the Riemann method is given below.

If $M[w]$ is an operator adjoint to $L[x]$, then the following well known identity is valid:

$$xM[w] - wL[x] = \frac{\partial}{\partial r}\left[x\left(\frac{\partial w}{\partial s} + \frac{nw}{r+s}\right)\right] - \frac{\partial}{\partial s}\left[w\left(\frac{\partial x}{\partial r} - \frac{nx}{r+s}\right)\right] \tag{3.10}$$

Let $F(r_0, s_0)$ be an arbitrary point on the curve *BE*. Let us impose the conditions

$$M[w] = 0, \qquad \frac{\partial w}{\partial s} + \frac{nw}{r_0+s} = 0 \quad \text{on } FG \tag{3.11}$$

on the function w.

The function

$$w = \frac{(s+r_0)^n (r-r_0)^n}{(r+s)^{2n}} \tag{3.12}$$

satisfies conditions (3.11), for example.

Let us integrate (3.10) over the area of the curvilinear trapezoid *ABFG* and then let us use the Green's formula and the relations (3.1), (3.7), (3.11):

$$\int_{AB} x\left(\frac{\partial w}{\partial s} + \frac{nw}{r+s}\right) ds + \int_{BF} w\,\frac{\partial x}{\partial r}\,dr + \int_{GA} w\left(\frac{\partial x}{\partial r} - \frac{nx}{r+s}\right) dr = 0 \qquad (3.13)$$

Substituting the values of w and x from (3.4), (3.9) and (3.12 into (3.13) and using $s = r - v$, we obtain the nonlinear integral equation

$$\int_{1/2}^{r_*} \frac{(r-v+r_0)^n (r-r_0)^n}{(2r-v)^{2n+1}}\,\frac{dv}{dr}\,dr = \mu\left(\frac{1}{2}+r_0\right)^n\left(\frac{1}{2}-r_0\right)^n +$$

$$+\int_{1/2}^{s_*} \frac{(s+r_0)^n (1/2-r_0)^n}{(1/2+s)^{2n+1}}\,ds - (1+2n\mu)\int_{1/2}^{r_0} \frac{(1/2+r_0)^n (r-r_0)^n}{(1/2+r)^{2n+1}}\,dr \qquad (3.14)$$

$$s_* = 1/2 - v_*, \qquad 1/2 - v_* \leqslant r_0 \leqslant 1/2$$

Hence, the piston velocity $v = v(\tau)$ is determined by using (3.8).

When n is a positive integer equation (3.14) is reduced by means of $2n + 1$ differentiations with respect to r_0 to the nonlinear differential equation

$$\frac{d^n}{dr^n}\left[\frac{1}{(2r-v)^{n+1}}\,\frac{dv}{dr}\right] + \sum_{m=n+1}^{2n-1} \frac{m!}{(m-n)!\,(2n-m)!}\,\frac{d^{2n-m}}{dr^{2n-m}}\left[\frac{1}{(2r-v)^{m+1}}\,\frac{dv}{dr}\right] +$$

$$+\frac{(2n)!}{n!}\,\frac{1}{(2r-v)^{2n+1}}\,\frac{dv}{dr} = 0 \qquad (3.15)$$

(the subscripts on the r_0 are omitted here). The initial data for equation (3.15) are determined easily in the process of successive differentiation of (3.14). Later we shall write them down for the values $n = 1;\ 2$. If the solution of (3.15) has been found, then the values of x, $\partial x/\partial r$, $\partial x/\partial s$ are determined from the relations (3.7), (3.9) on BE. Then we have a Cauchy problem for equation (3.1) in the triangle BEC, which can be solved by the Riemann method [6]. Later we shall be satisfied with finding the piston velocity v.

4. For $n = 1$ ($\gamma = 3$), equations (3.15), (3.8) have the form

$$\frac{d}{dr}\left[\frac{1}{(2r-v)^2}\,\frac{dv}{dr}\right] + \frac{2}{(2r-v)^3}\,\frac{dv}{dr} = 0, \qquad \frac{dv}{d\tau} = (2r-v)^3 \qquad (4.1)$$

The initial conditions for these equations are

$$v\left(\frac{1}{2}\right) = \frac{\mu}{1+\mu}, \qquad \left(\frac{dv}{dr}\right)_{r=1/2} = -1, \qquad \tau\left(\frac{1}{2}\right) = \frac{\mu(\mu+2)}{2} \qquad (4.2)$$

Equation (4.1) is invariant with respect to the two-parameter group of transformations:

$$v = \alpha v' + \beta, \qquad r = \alpha r' + 1/2\beta \qquad (4.3)$$

This circumstance affords the possibility of integrating (4.1) in elementary functions. Let us present the final result for the piston velocity as a function of time

$$v=\frac{1}{1+\mu}\left[3+\mu-\sqrt{\frac{4\tau+1-2\mu-\mu^2}{2\tau+1}}-2\sqrt{\frac{2\tau+1}{4\tau+1-2\mu-\mu^2}}\right]$$

Staniukovich obtained the solution for this case by another method [2].

5. For the case $n = 2$ ($\gamma = 1.67$), equations (3.15), (3.8) have the form

$$\frac{d^2}{dr^2}\left[\frac{1}{(2r-v)^3}\frac{dv}{dr}\right]+6\frac{d}{dr}\left[\frac{1}{(2r-v)^4}\frac{dv}{dr}\right]+\frac{12}{(2r-v)^5}\frac{dv}{dr}=0, \qquad \frac{dv}{d\tau}=(2r-v)^5 \tag{5.1}$$

The initial conditions for these equations will be

$$v\left(\frac{1}{2}\right)=1-\frac{1}{\sqrt{1+2\mu}}, \qquad \left(\frac{dv}{dr}\right)_{r=1/2}=\frac{1}{\sqrt{1+2\mu}}-2$$
$$\left(\frac{d^2v}{dr^2}\right)_{r=1/2}=-15\sqrt{1+2\mu}-\frac{3}{\sqrt{1+2\mu}}+12, \qquad \tau\left(\frac{1}{2}\right)=\mu\,(1+\mu) \tag{5.2}$$

The first equation of (5.1) is invariant with respect to the groups (4.3). Hence the order of the equation can be reduced by two. To do this let us make the change of variable:

$$2r-v=e^{\varphi}, \qquad dv\,/\,dr=y, \qquad dy\,/\,d\varphi=z \tag{5.3}$$

After the change of variable (5.3), we obtain an Abel equation of the second kind:

$$12y\,(1-y)^2+(2-y)\,(10y-8)\,z-(2-y)z^2+(2-y)^2z\,dz\,/\,dy=0 \tag{5.4}$$

Furthermore, the substitution $q = z(2 - y)$ yields

$$12y\,(1-y)^2+2\,(5y-4)\,q+q\,dq\,/\,dy=0 \tag{5.5}$$

Equation (5.5) has the particular solution $q_1 = -2y^2 + 2y$. After substituting $q = \psi + q_1$, we obtain

$$6\,(y-1)\,\psi+[\psi-2y\,(y-1)]\,d\psi\,/\,dy=0 \tag{5.6}$$

By the change of variables

$$\psi=-2\left(\xi+\frac{1}{2}\eta^2\right)^3, \qquad y=-\eta\left(\xi+\frac{1}{2}\eta^2\right)$$

(5.6) is reduced to a special Riccati equation:

$$\frac{d\eta}{d\xi} - \frac{1}{2}\eta^2 = \xi \tag{5.7}$$

Equation (5.7) cannot be integrated in terms of elementary functions or quadratures for the given right side (Liouville theorem [9]). The

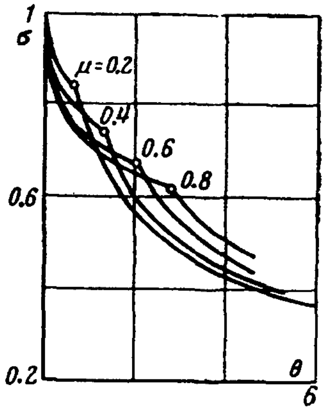

Fig. 3.

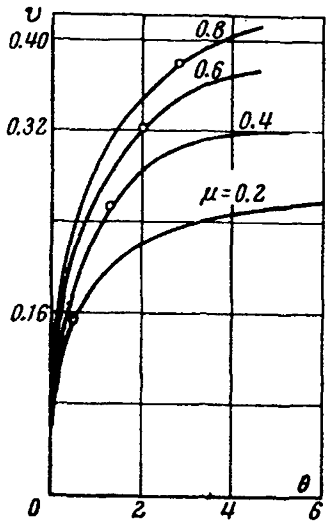

Fig. 4.

solution of (5.7) expressed in terms of Bessel functions (Schaffli solution [9]) has the form

$$\varepsilon(\lambda) = \frac{cJ_{2/3}(\lambda) + J_{-2/3}(\lambda)}{cJ_{-1/3}(\lambda) - J_{1/3}(\lambda)}, \quad \varepsilon = \frac{\eta}{\sqrt{2\xi}}, \quad \lambda = \frac{\sqrt{2}}{3}\xi^{3/2} \quad c = \text{const} \tag{5.8}$$

Let ε_*, λ_* denote the values of ε and λ for $r = 1/2$. Using the initial data (5.2) and the transformation formulas relating r, v to the variables ε, λ we obtain

$$\varepsilon_* = \frac{2 - (1 + 2\mu)^{-1/2}}{[2(1 + 2\mu)^{-1/2} - 1]^{1/2}}, \qquad \lambda_* = \frac{[2(1 + 2\mu)^{-1/2} - 1]^{3/2}}{9 + 3(1 + 2\mu)^{-1} - 6(1 + 2\mu)^{-1/2}} \tag{5.9}$$

The following expression for the arbitrary constant can be found from (5.8) and (5.9):

$$c = \frac{\varepsilon_* J_{1/3}(\lambda_*) + J_{-2/3}(\lambda_*)}{\varepsilon_* J_{-1/3}(\lambda_*) - J_{2/3}(\lambda_*)} \tag{5.10}$$

Returning to the v, τ variables and using (5.2) we obtain the piston velocity, the time at the piston and the speed of sound at the piston as a function of λ

$$v(\lambda) = v_* - \int_{\lambda_*}^{\lambda} \sigma(\lambda)\, \varepsilon(\lambda)\, d\lambda, \quad \tau(\lambda) = \tau_* - \int_{\lambda_*}^{\lambda} \frac{\varepsilon(\lambda)\, d\lambda}{\sigma^4(\lambda)} \tag{5.11}$$

$$\sigma(\lambda) = \sigma_* \exp\left[\int_{\lambda_*}^{\lambda} \varepsilon(\lambda)\, d\lambda + \frac{2}{3}\int_{\lambda_*}^{\lambda} \frac{d\lambda}{\lambda(\varepsilon^2 + 1)}\right]$$
$$\sigma_* = 1 - v_* \tag{5.12}$$

From the definitions of v and θ we have

$$\frac{d\theta}{d\tau} = \frac{4n-2}{\mu} v, \qquad \theta = \theta_* + \frac{6}{\mu}\int_{\lambda_*}^{\lambda} v \frac{d\tau}{d\lambda}\, d\lambda \tag{5.13}$$

Let us introduce the notation

$$\begin{aligned} A(\lambda) &= cJ_{2/3}(\lambda) + J_{-2/3}(\lambda), \qquad A_* = A(\lambda_*) \\ B(\lambda) &= cJ_{-1/3}(\lambda) - J_{1/3}(\lambda), \qquad B_* = B(\lambda_*) \end{aligned} \tag{5.14}$$

From the recursion formulas for the Bessel functions we easily obtain [9]

$$A = -\frac{1}{3\lambda} B - \frac{dB}{d\lambda}, \qquad B = \frac{2}{3\lambda} A + \frac{dA}{d\lambda} \tag{5.15}$$

Substituting (5.14) into (5.12), we find

$$\sigma(\lambda) = \sigma_* \exp \int_{\lambda_*}^{\lambda} \left[\frac{A}{B} + \frac{2}{3\lambda}\frac{B^2}{A^2 + B^2}\right] d\lambda \tag{5.16}$$

Using (5.15), we have

$$\frac{d}{d\lambda}(A^2 + B^2) = -\frac{2}{3\lambda}[2A^2 + B^2] \tag{5.17}$$

Let us represent the equality (5.16) in the form

$$\sigma(\lambda) = \sigma_* \exp \int_{\lambda_*}^{\lambda} \left[-\frac{1}{3\lambda} - \frac{1}{B}\frac{dB}{d\lambda} + \frac{2}{3\lambda}\frac{B^2}{A^2 + B^2} - \frac{4}{3\lambda} + \frac{4}{3\lambda}\right] d\lambda$$

Hence

$$\sigma(\lambda) = \sigma_* \exp \int_{\lambda_*}^{\lambda} \left[\frac{1}{\lambda} - \frac{1}{B}\frac{dB}{d\lambda} - \frac{2}{3\lambda}\frac{2A^2 + B^2}{A^2 + B^2}\right] d\lambda \tag{5.18}$$

Integrating (5.18), taking account of (5.17), we obtain the speed of sound at the piston

$$\sigma(\lambda) = \frac{\sigma_* B_*}{\lambda_* (A_*^2 + B_*^2)} \frac{\lambda [A^2(\lambda) + B^2(\lambda)]}{B(\lambda)} \tag{5.19}$$

In an analogous manner, we obtain from (5.11) to (5.15)

$$\tau(\lambda) = \tau_* + \frac{\lambda_*^4 (A_*^2 + B_*^2)^4}{\sigma_*^4 B_*^4} \left\{ \frac{1}{2} \frac{A(\lambda) B(\lambda)}{\lambda^3 [A^2(\lambda) + B^2(\lambda)]^3} + \frac{3}{4} \frac{1}{\lambda^2 [A^2(\lambda) + B^2(\lambda)]^2} - \right.$$
$$\left. - \frac{1}{2} \frac{A_* B_*}{\lambda_*^3 (A_*^2 + B_*^2)^3} - \frac{3}{4} \frac{1}{\lambda_*^2 (A_*^2 + B_*^2)^2} \right\} \tag{5.20}$$

$$v(\lambda) = 1 - \sigma(\lambda) - \frac{2}{3} \frac{\sigma_* B_*}{\lambda_* (A_*^2 + B_*^2)} \int_{\lambda_*}^{\lambda} B(\lambda)\, d\lambda = 1 - \sigma(\lambda) - \frac{4}{3} \frac{\sigma_* B_*}{\lambda_* (A_*^2 + B_*^2)} \times$$
$$\times \left[\sum_{k=0}^{\infty} J_{4/3+2k}(\lambda) - c \sum_{k=0}^{\infty} J_{2/3+2k}(\lambda) - \sum_{k=0}^{\infty} J_{4/3+2k}(\lambda_*) + c \sum_{k=0}^{\infty} J_{2/3+2k}(\lambda_*) \right]$$

$$\theta(\lambda) = \theta_* + \frac{6}{\mu} v(\tau - \tau_*) + \frac{3}{\mu} \frac{A_* (A_*^2 + B_*^2) \lambda_*}{\sigma_*^4 B_*^3} (v - v_*) + \frac{9}{2\mu} \frac{(A_*^2 + B_*^2) \lambda_*^2}{\sigma_*^4 B_*^4} \times$$
$$\times (v - v_*) + \frac{9}{2\mu} \frac{\lambda_*^3 (A_*^2 + B_*^2)^3}{\sigma_*^3 B_*^3} \left\{ \frac{1}{\lambda B(\lambda) [A^2(\lambda) + B^2(\lambda)]} - \frac{1}{\lambda_* B_* (A_*^2 + B_*^2)} \right\}$$

Formulas (5.19) and (5.20) afford the possibility of determining the piston velocity and the speed of sound at the piston in the first reflection zone. Examples of the numerical computations are given in Figs. 3 and 4. (The open circles are the ends of the simple waves.)

BIBLIOGRAPHY

1. Love, A.E.H. and Pidduck, F.B., Lagrange's ballistic problem. *Trans. Roy. Soc.* London, 222, 167-226, 1922.

2. Staniukovich, K.P., *Neustanovivshiesia dvizheniia sploshnoi sredy (Unsteady Motion of Continuous Media)*. Gostekhizdat, 1955.

3. Courant, R. and Friedrichs, K., *Sverkhzvukovoe techenie i udarnye volny (Supersonic Flow and Shock Waves)*. IIL, 1950.

4. Landau, L.D. and Lifshits, E.M., *Mekhanika sploshnykh sred (Continuum Mechanics)*. Gostekhizdat, 1954.

5. Mises, R., *Matematicheskaia teoriia techenii szhimaemoi zhidkosti (Mathematical Theory of Compressible Fluid Flow)*. IL, 1961.

6. Riemann, B., *O rasprostranenii ploskikh voln s konechnoi amplitudoi (On the propagation of plane waves of finite amplitude)*. Sochineniia (Collection). Gostekhizdat, 1948.

7. Goursat, E., *Kurs matematicheskogo analiza (Course in Mathematical Analysis)*. Vol. 3, Pt. 1. GTTI, 1933.

8. Hadamard, J., Sur probleme mixte aux dérivées partielles. *Bull. Soc. Mate. France*, Vol. 31, 1933.

9. Watson, G.N., *Teoriia besselevykh funktsii (Theory of Bessel Functions)*. Pt. 1. IL, 1949.

Translated by M.D.F.